Mathematical Induction

Part One

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$$\boldsymbol{P(0)}\ \textbf{is true}$$

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

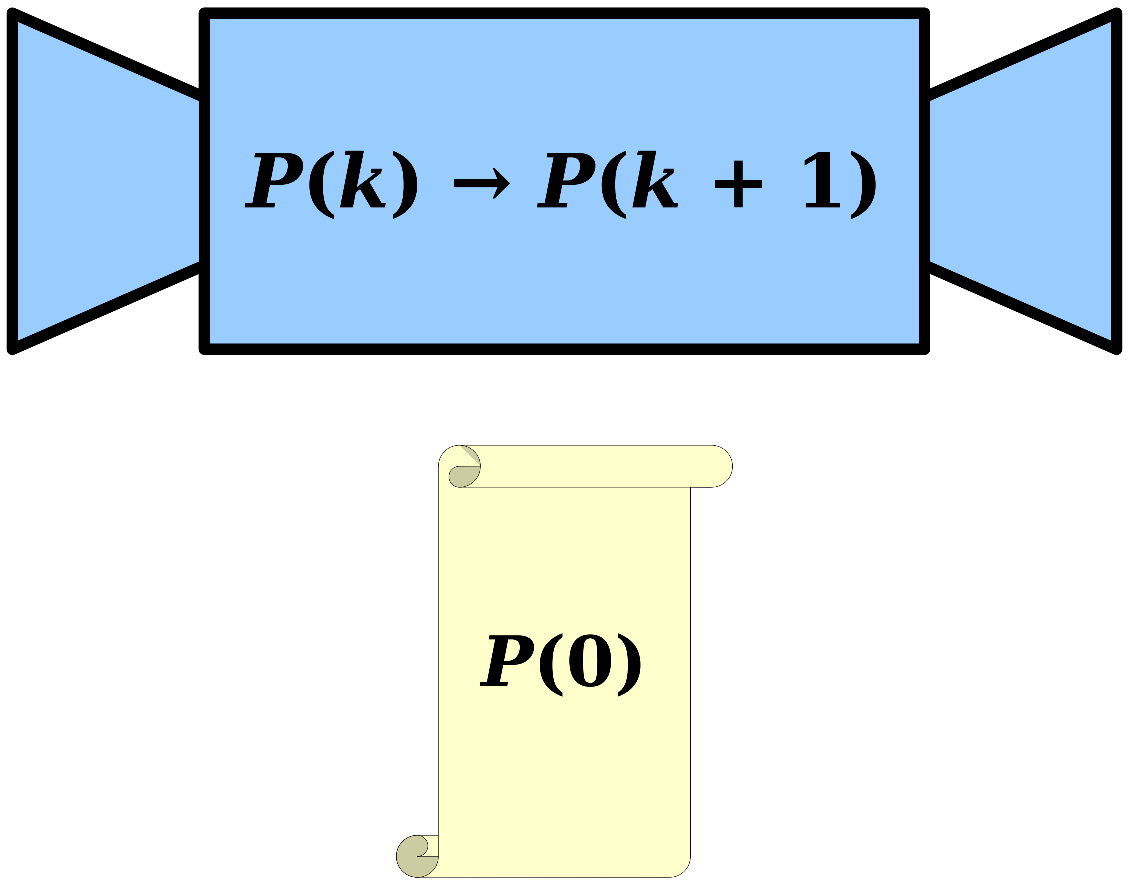
Induction, Intuitively

$$P(0)$$

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Why Induction Works



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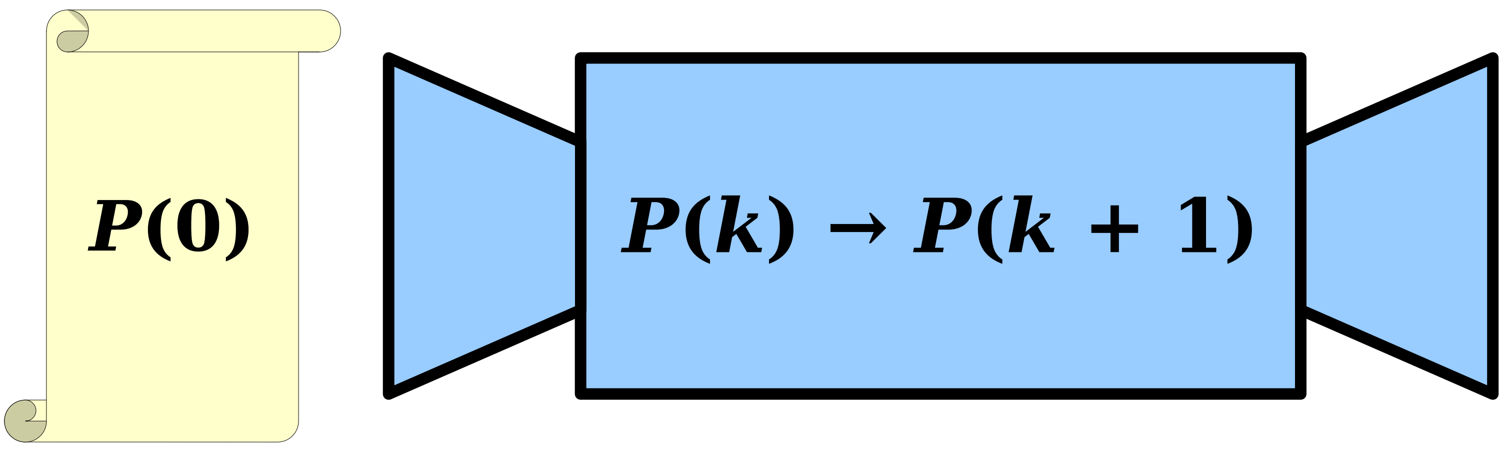

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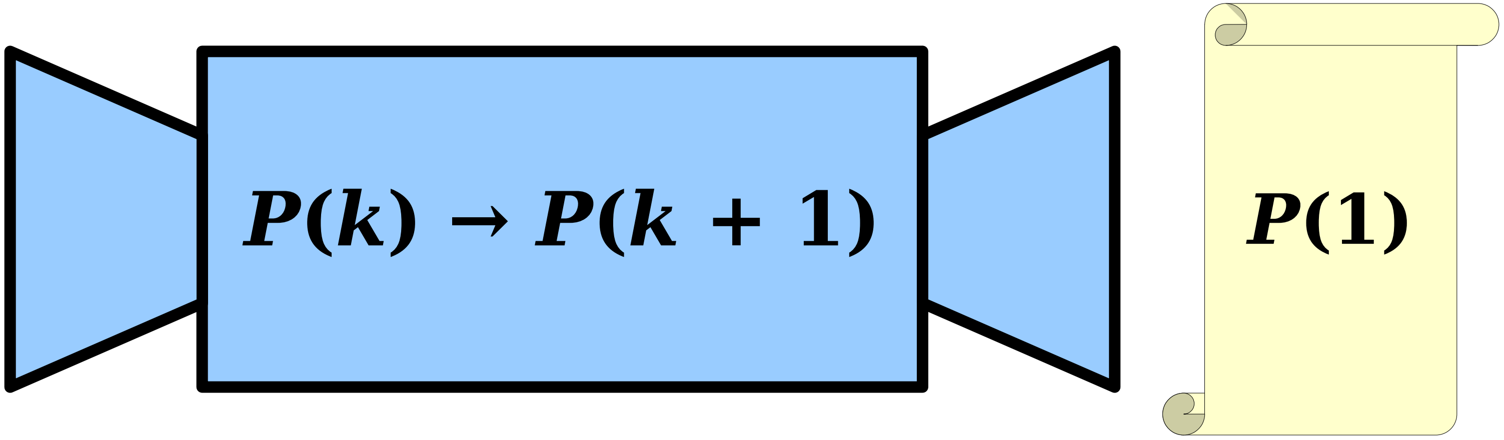

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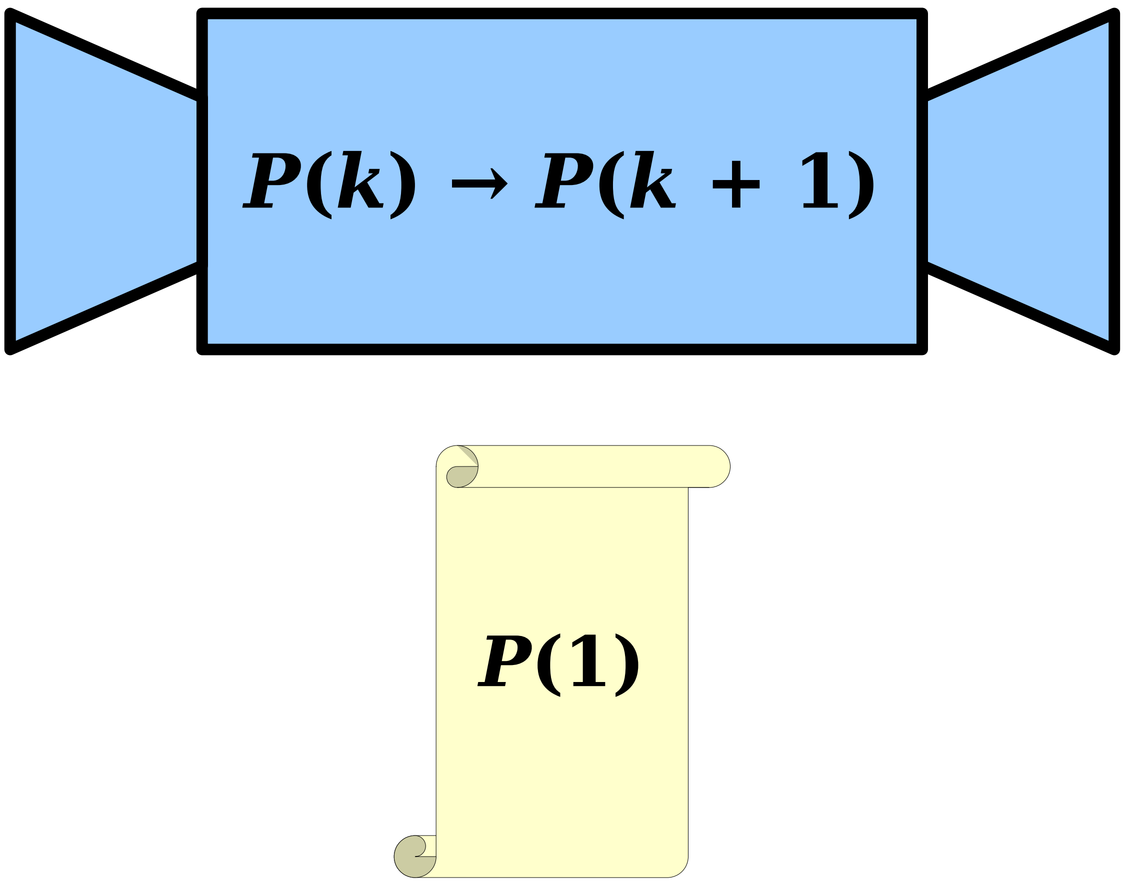

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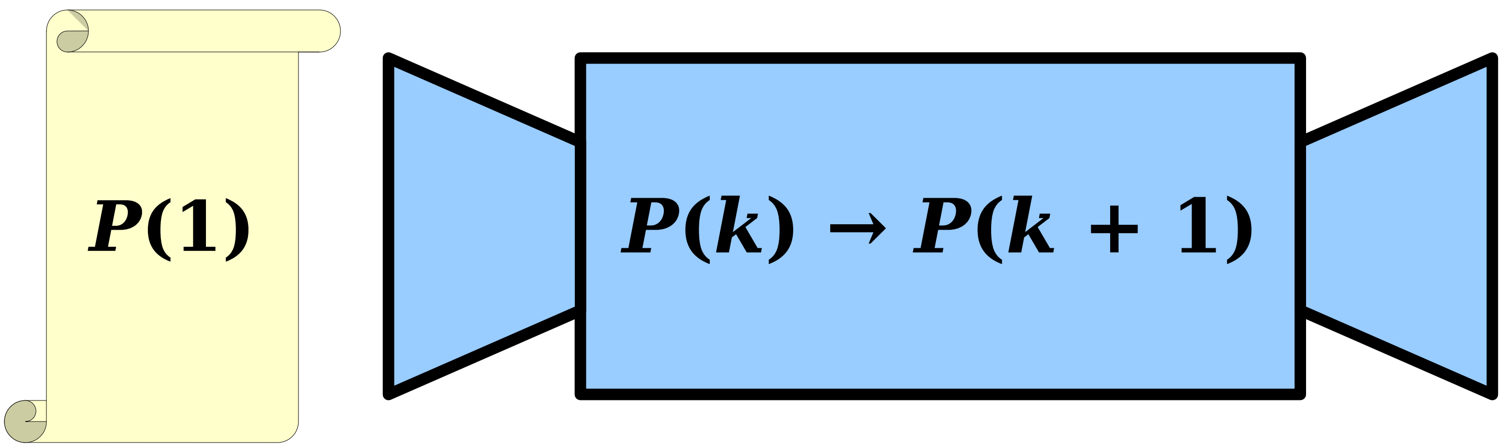

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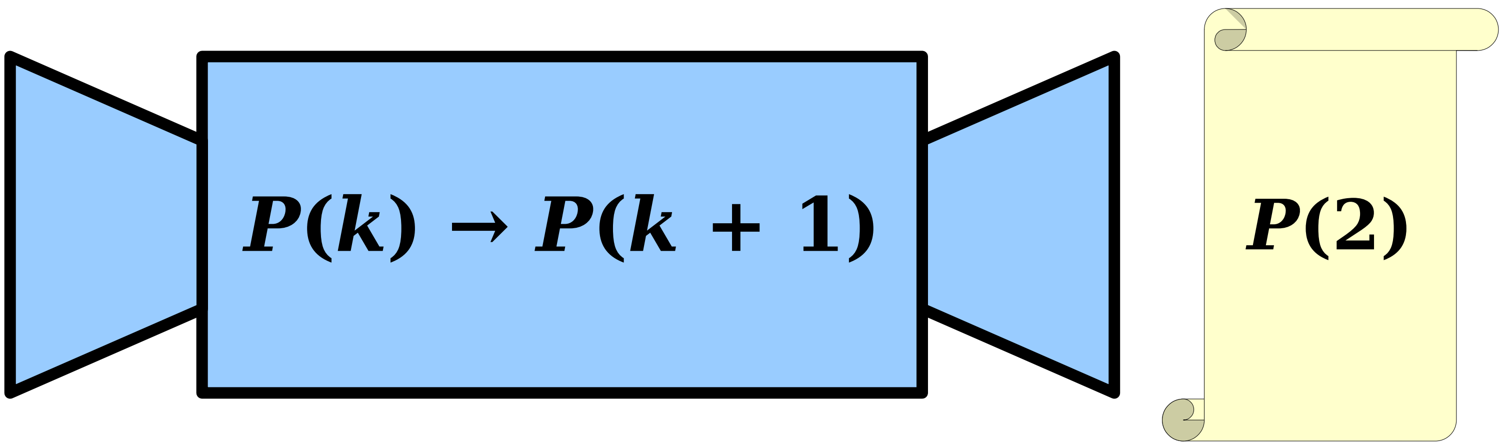

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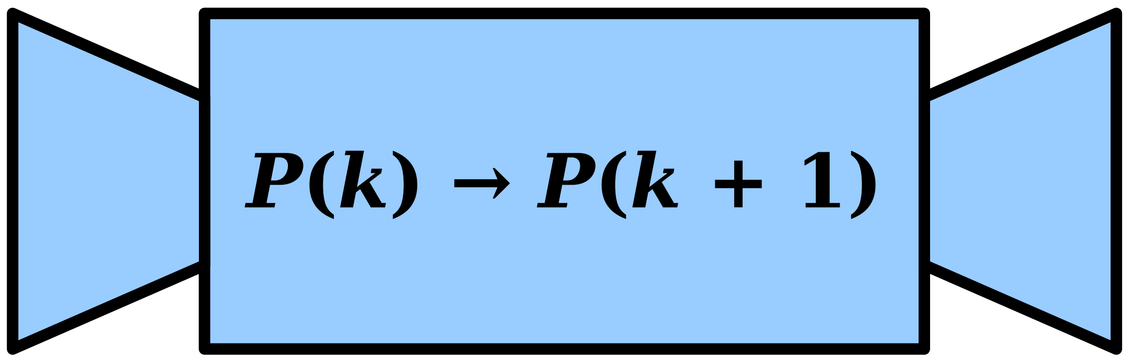

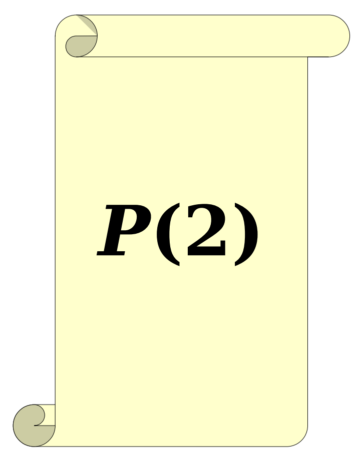

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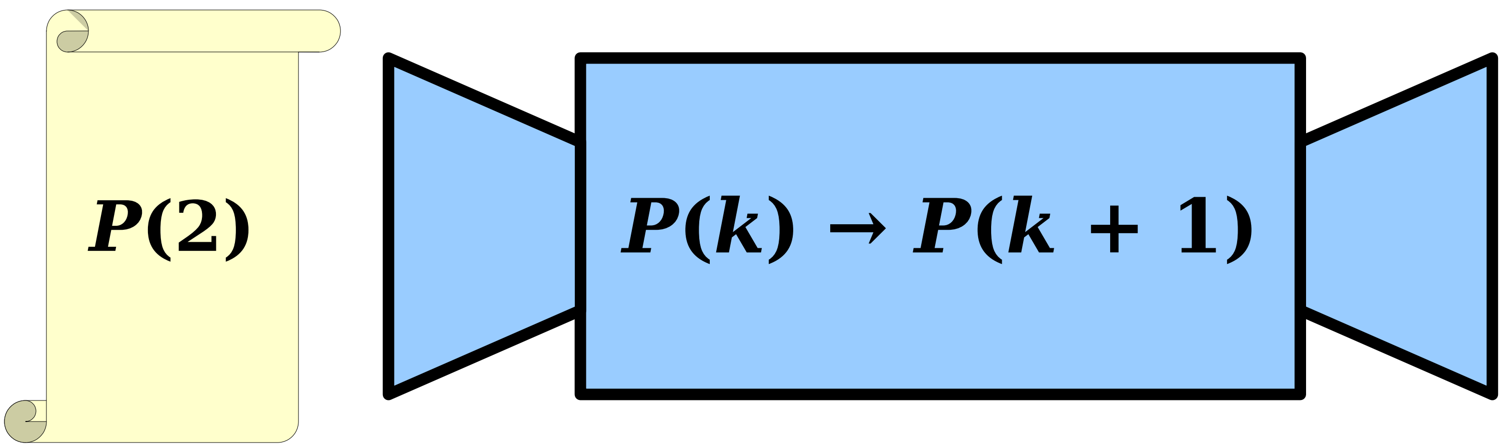

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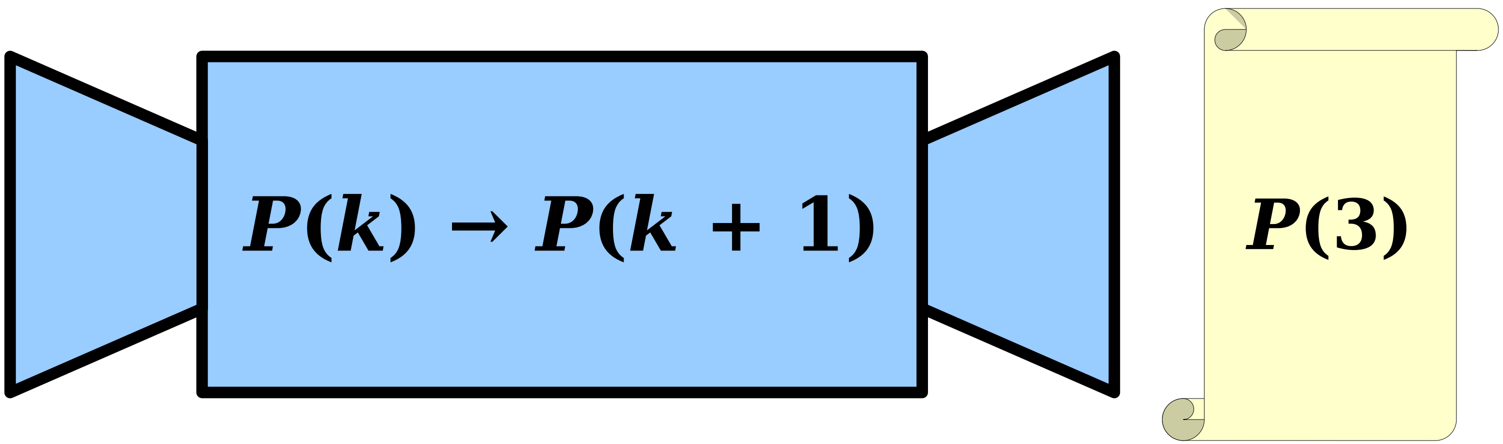

Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that $P(0)$ is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if $P(k)$ is true, then $P(k+1)$ is true.
    - This is called the ***inductive step***.
    - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \quad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

Theorem: The sum of the first n powers of two is $2^n - 1$.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers *n*, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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What should the next step of this proof be?

A. Prove that, for any k ∈ ℕ, that P(k) is true.
B. Assume for any k ∈ ℕ that P(k) and P(k+1) are true.
C. Assume that P(k) holds for all k ∈ ℕ.
D. Pick an arbitrary k ∈ ℕ, and prove P(k+1).
E. Pick an arbitrary k ∈ ℕ, assume P(k), and prove P(k+1).
F. None of these, or more than one of these.

Answer at pollev.com/zhenglian

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to prove it.

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Here, we'll use our **inductive hypothesis** (the assumption that *P*(*k*) is true) to simplify a complex expression. This is a common theme in inductive proofs.

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

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$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

A Quick Aside

- This result helps explain the range of numbers that can be stored in an **`int`**.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. If we have time, we'll see one today.

Structuring a Proof by Induction

- Define some predicate P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that $P(0)$ is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming $P(k)$ for some arbitrary natural number k, then write out exactly what that means.
  - Say that you're going to prove $P(k+1)$, then write out exactly what that means.
  - Prove that $P(k+1)$ using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

The Counterfeit Coin Problem

Problem Statement

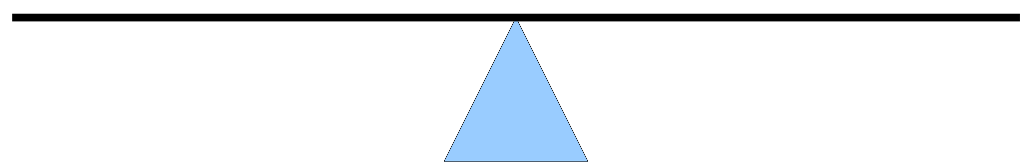
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

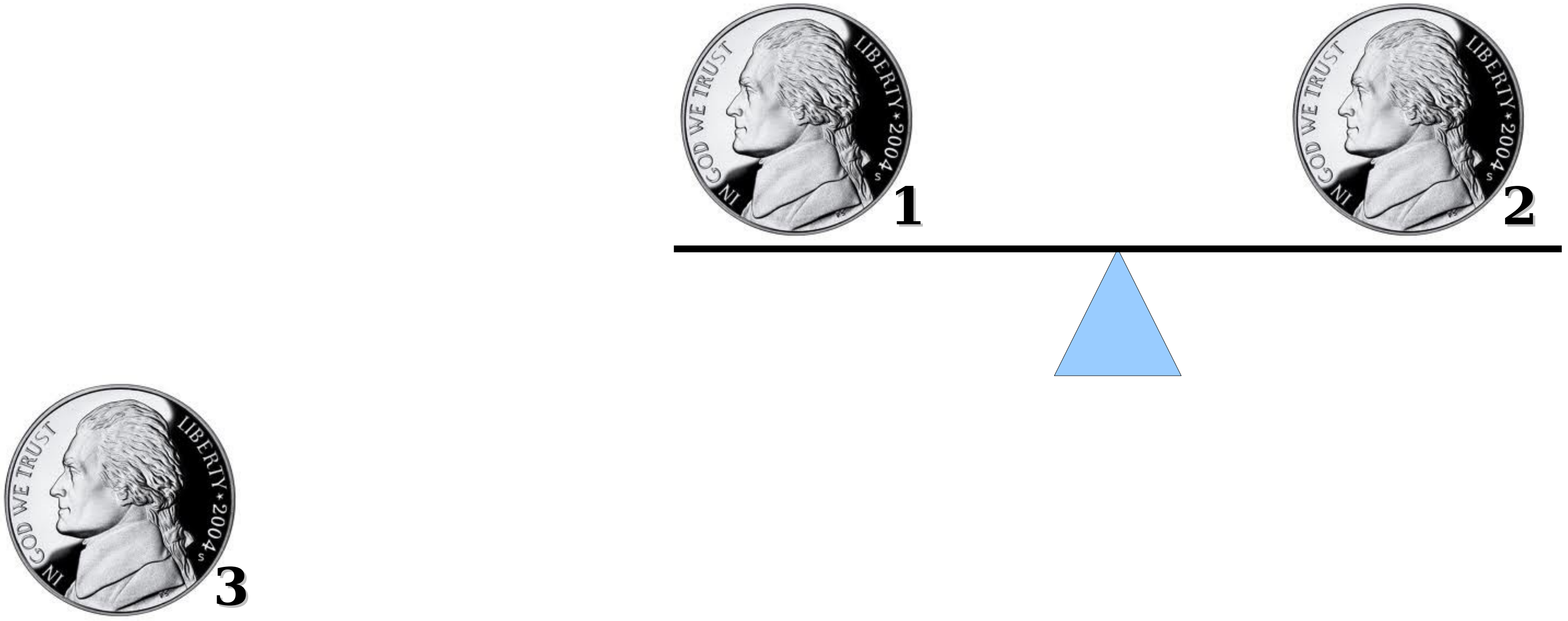
1

2

3



Finding the Counterfeit Coin



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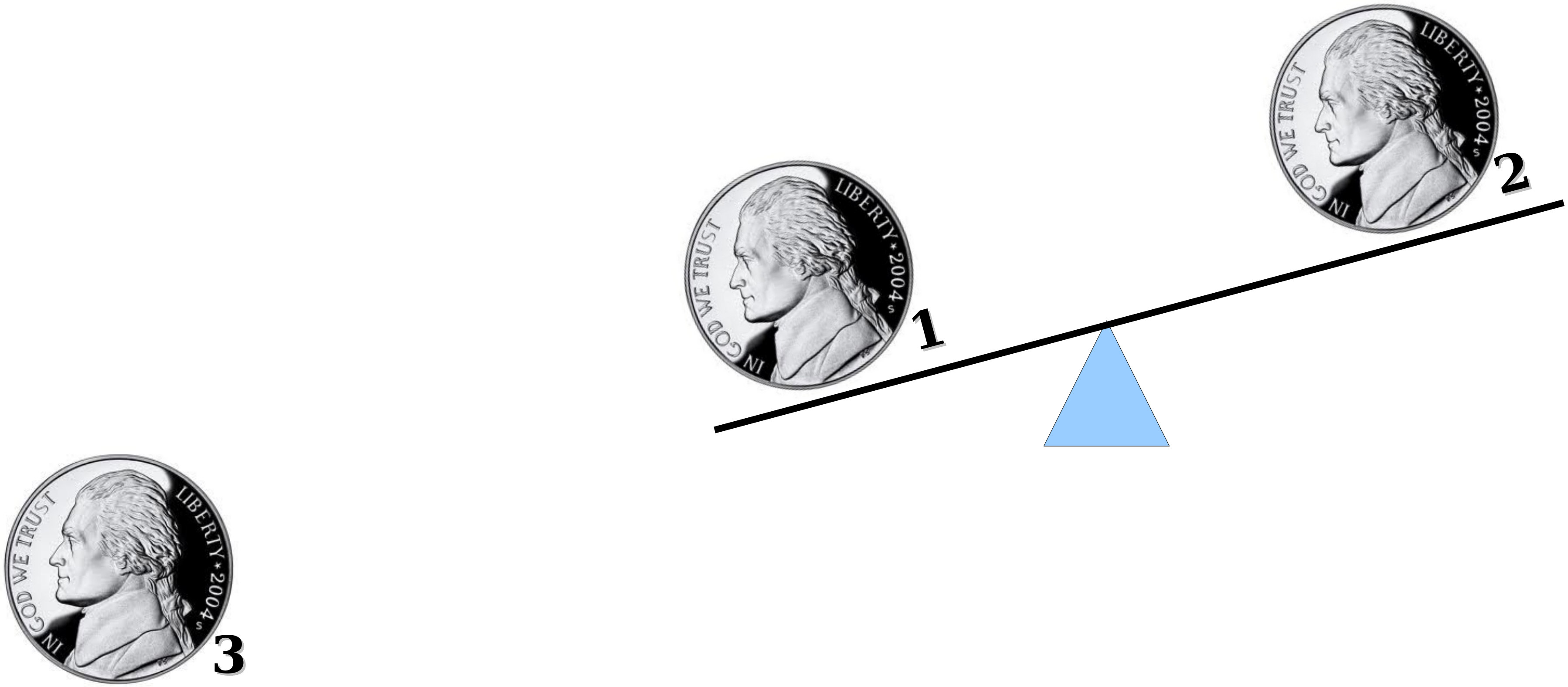

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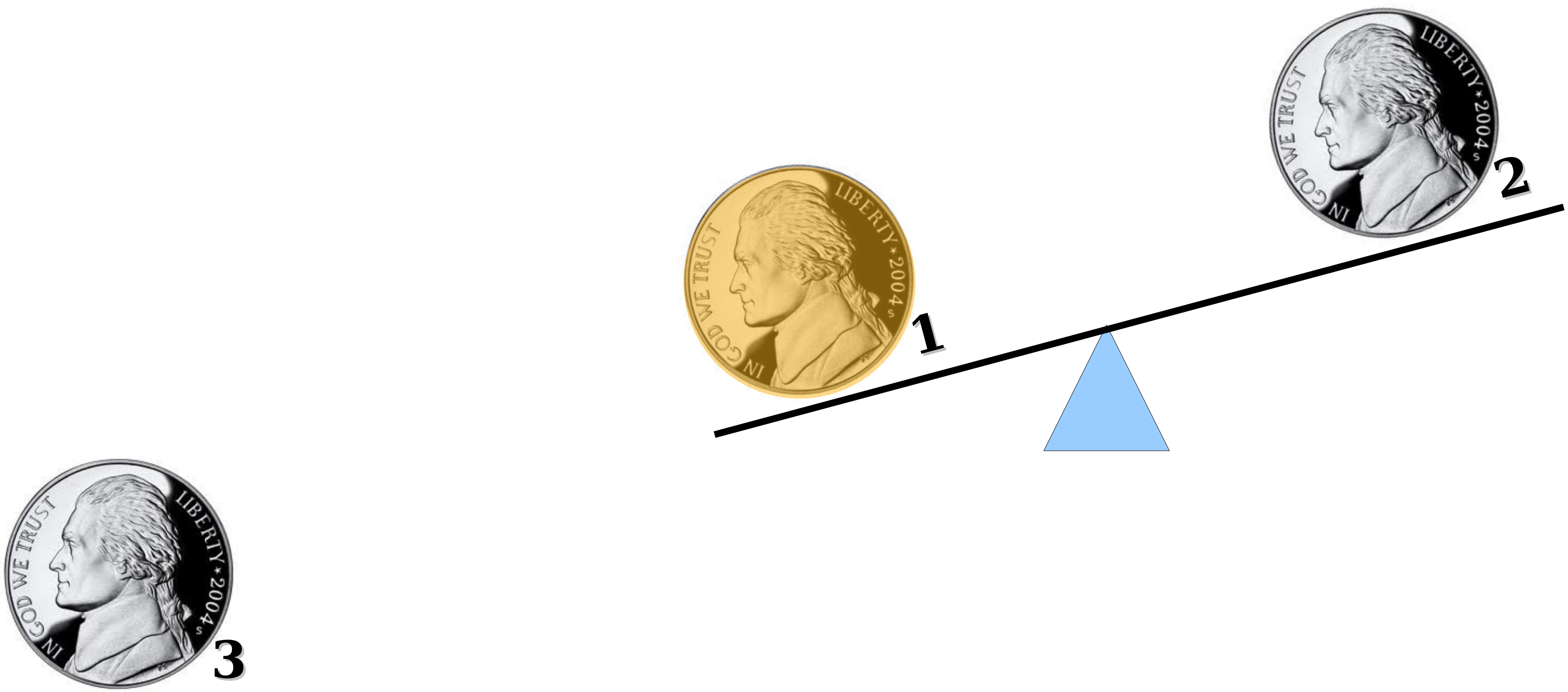

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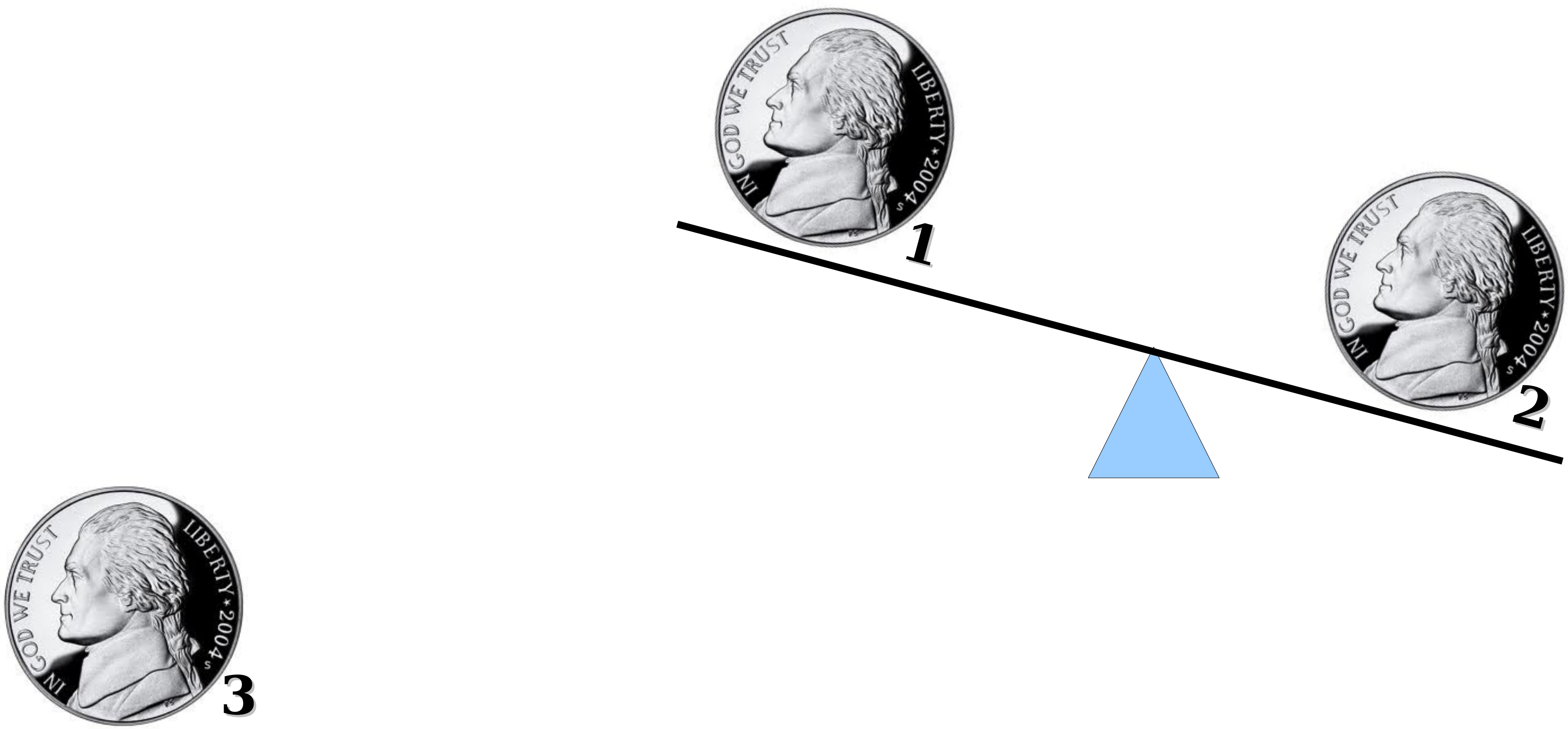

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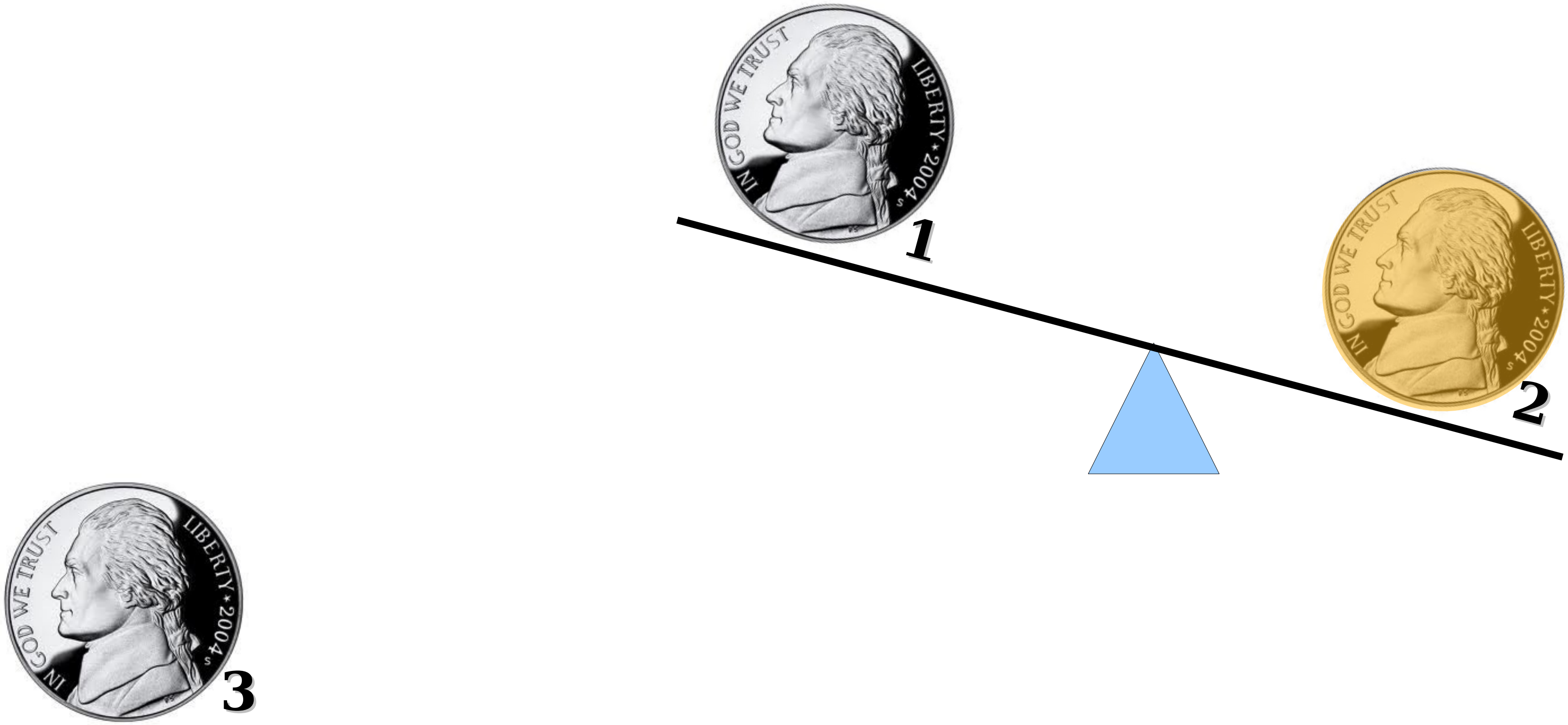

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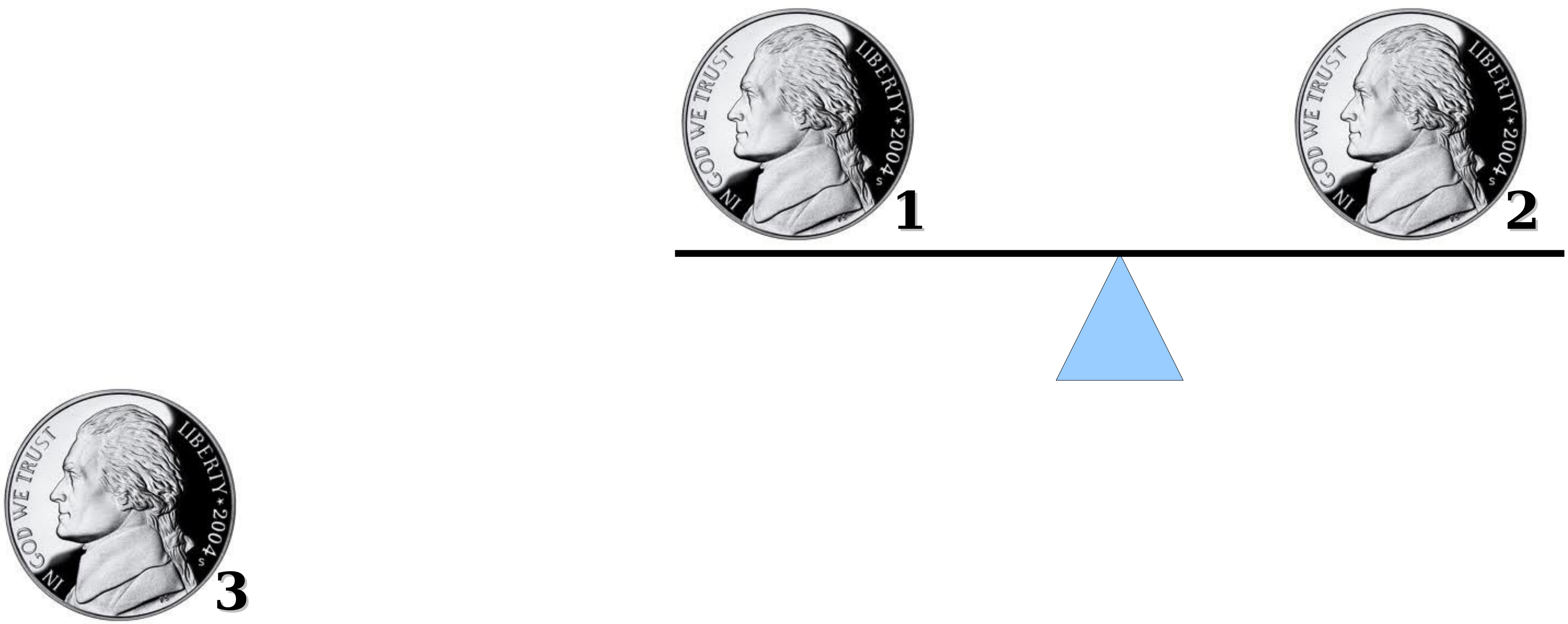

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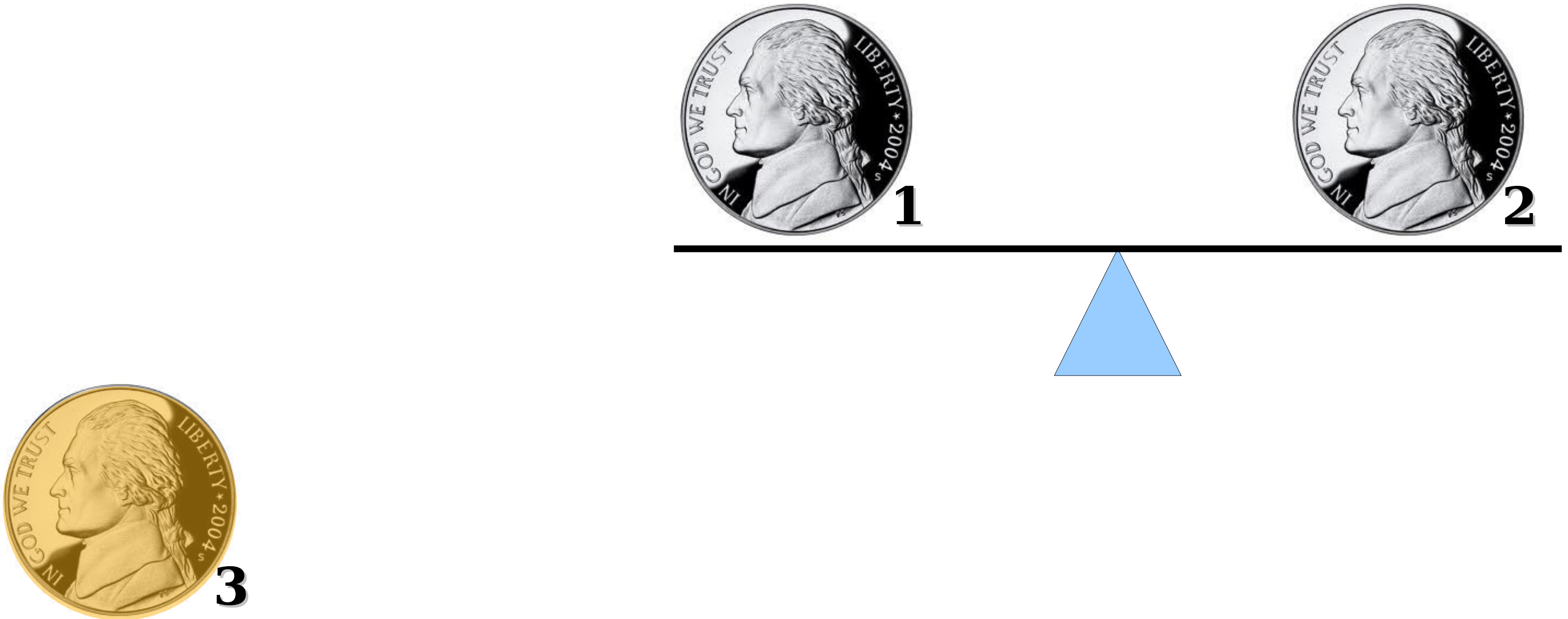
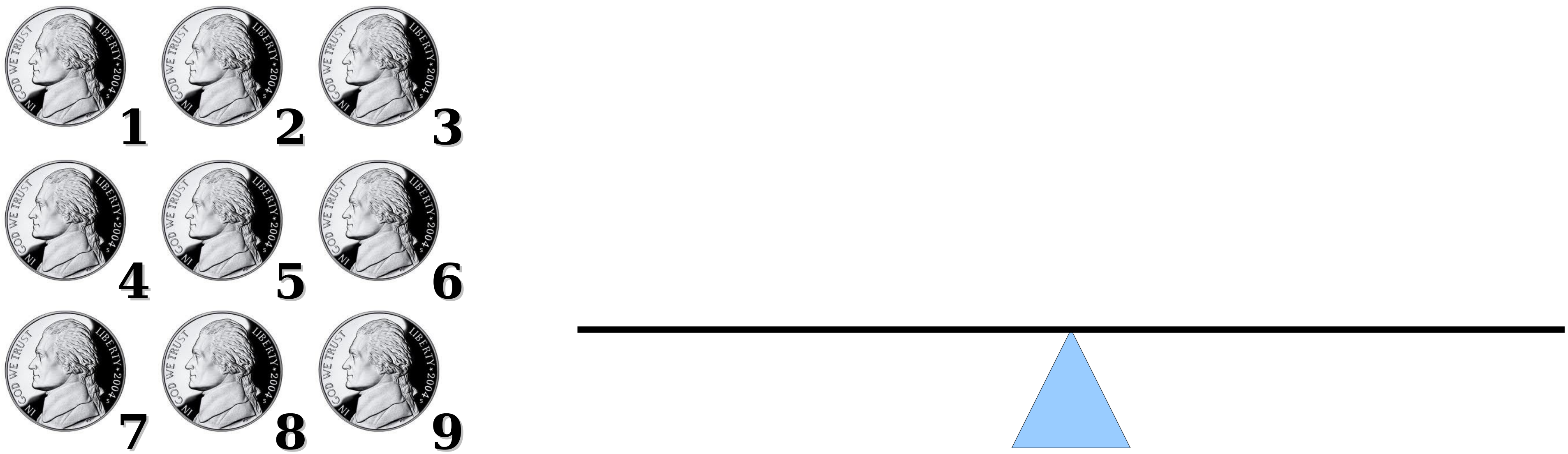


A Harder Problem

- You are given a set of ***nine*** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only ***two*** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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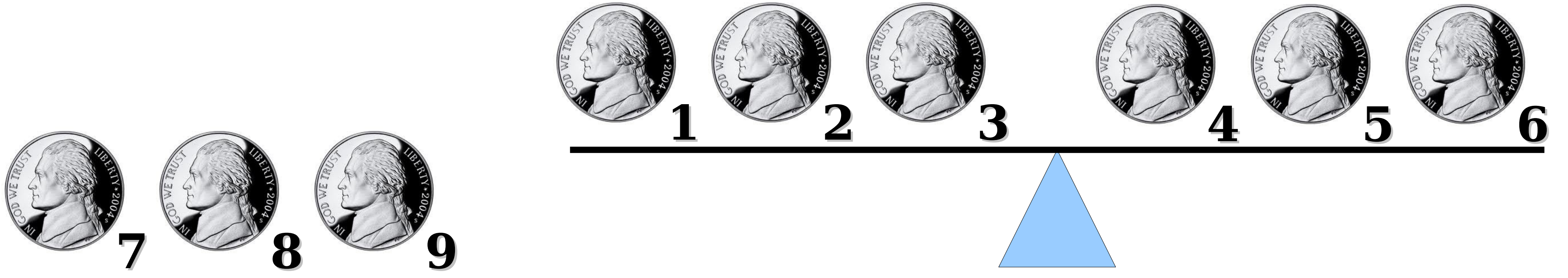

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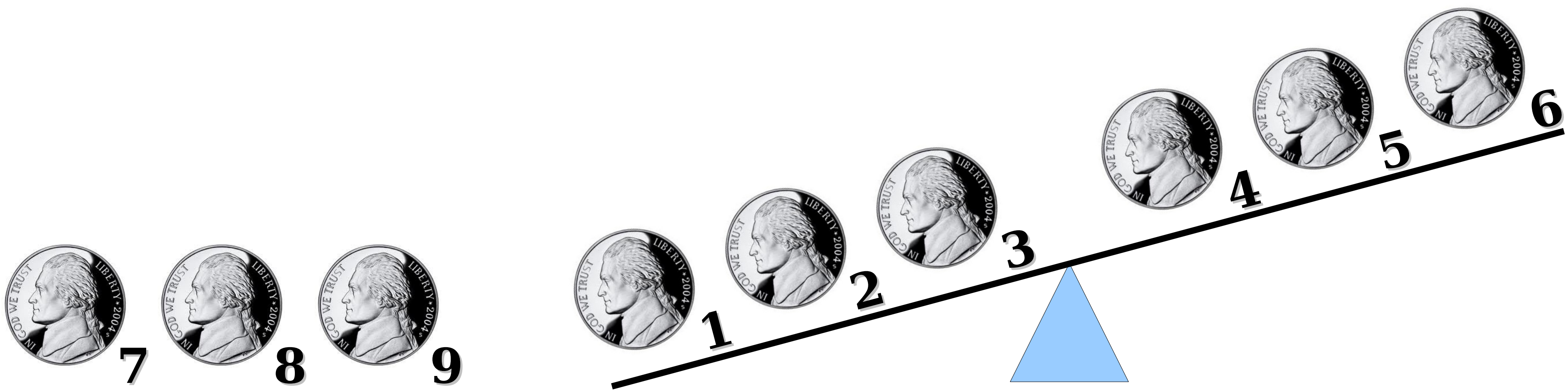

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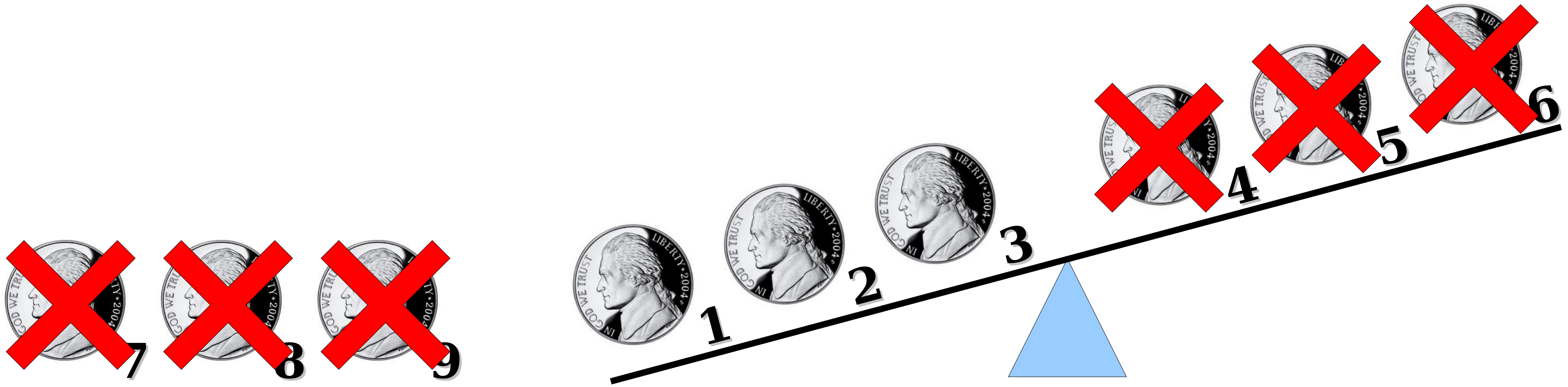

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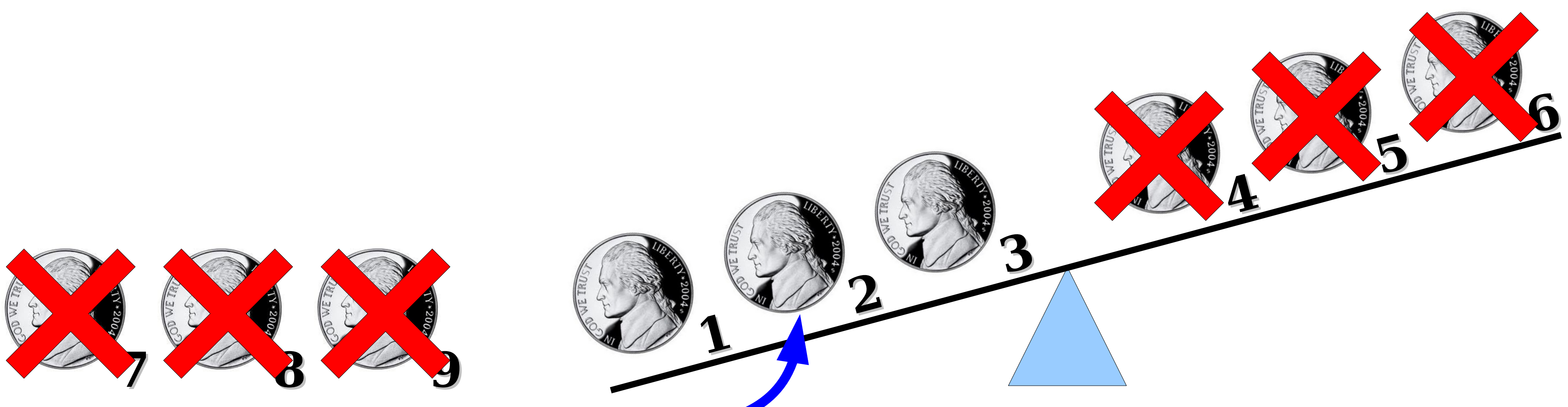
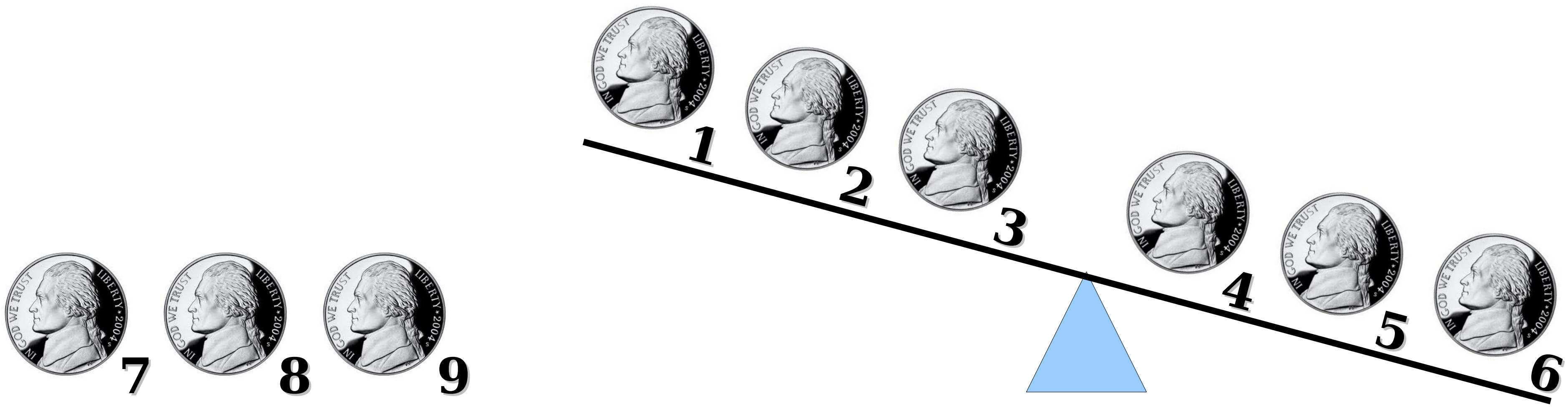


Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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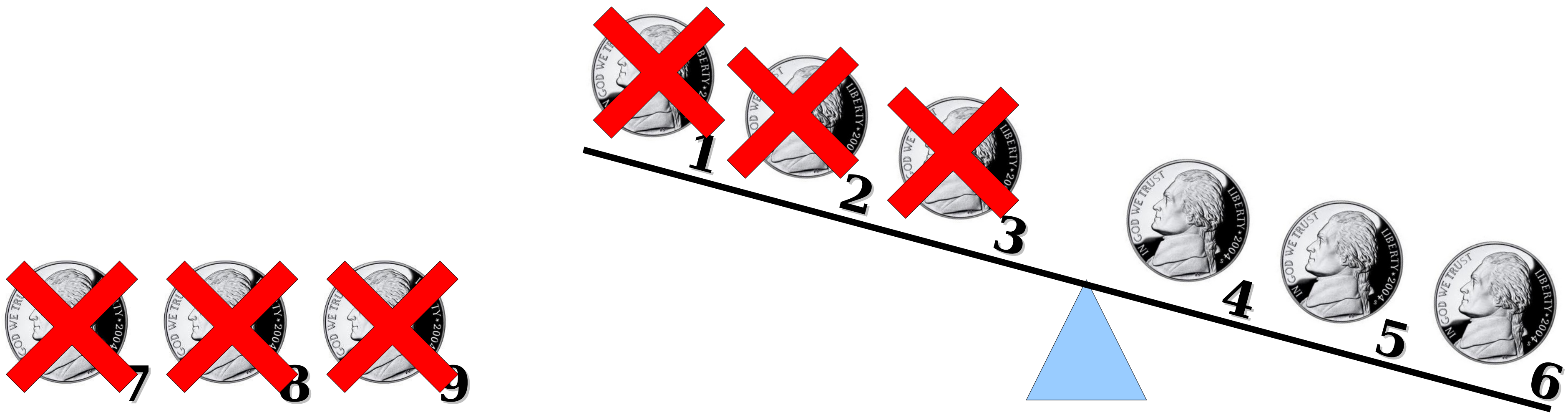

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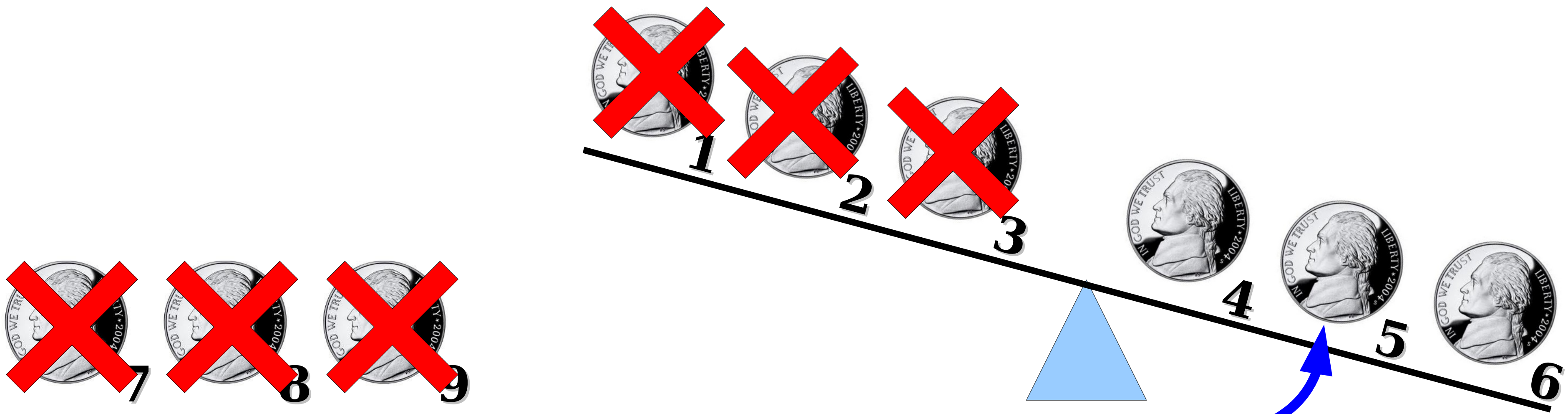

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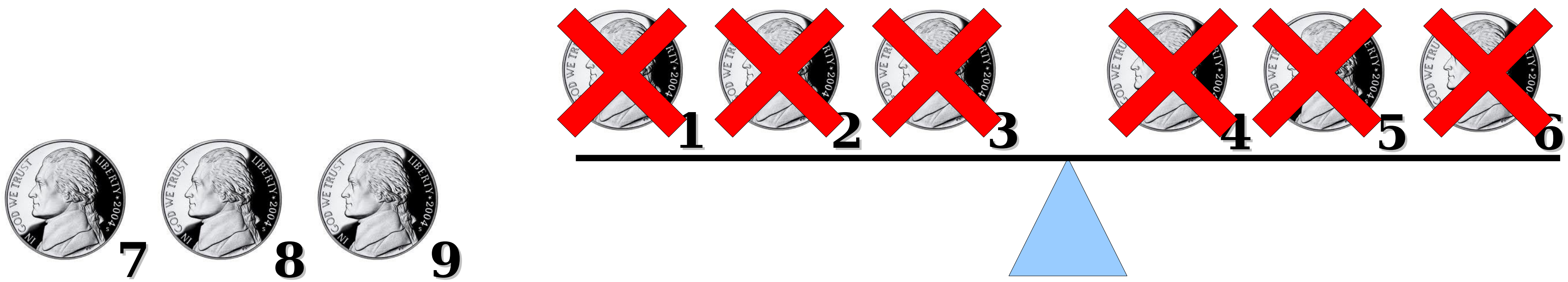

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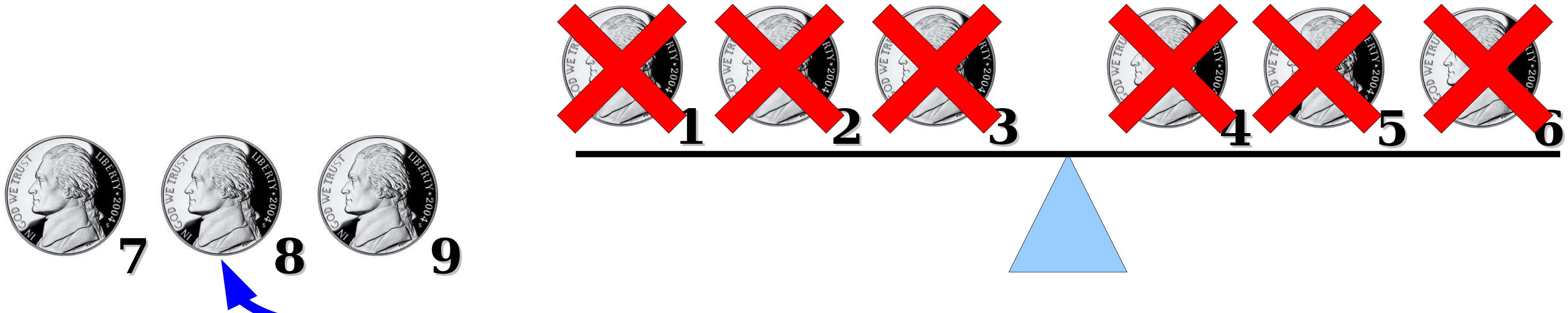

Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - ***One*** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of ***three*** coins.
- If we have two weighings, we can find the counterfeit out of ***nine*** coins.

So far, we have

$$\mathbf{1, 3, 9 = 3^0, 3^1, 3^2}$$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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In a proof by induction, we need to prove that

- ☐ $P(0)$ is true
- ☐ If $P(k)$ is true, then $P(k+1)$ is true.

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another.

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Proof: Let $P(n)$ be the following statement:

If exact … rest,
that c … ce.

Here, we use our **inductive hypothesis** (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

We'll use … rom which the theore…

As our ba… if we have a set of 3… n find that coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

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Some Fun Problems

- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?
  - Can you find the counterfeit out of a group of more than 3^k coins with k weighings?
  - Can you find the counterfeit out of any group of at most 3^k coins with k weighings?

Time-Out for Announcements!

Midterm Logistics

- Reminder: Midterm is next Friday (7/25), 5-8 PM PST, Hewlett 201.
- Keller has sent out exam logistics for OAE students.
- Abhy is hosting a review session next Tuesday (7/22), 3-4 PM PST.

Problem Set Two Graded

- Your TAs have just finished grading PS2. Grades and feedback are now available on Gradescope.
- Distribution:
  - 75^{th} Percentile: 34/38
  - 50^{th} Percentile: 29/38
  - 25^{th} Percentile: 25/38
- As always, ***please review your feedback!*** Knowing where to improve is more important than just seeing a raw score.
- Did we make a mistake? Submit a regrade request between this Friday, 5:30 PM PST and next Friday, 5:30 PM PST.

Back to CS103!

How Not To Induct

Something's Wrong...

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Theorem: The sum of the first n powers of two is 2^n.

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What's wrong with this proof?

Answer at pollev.com/zhenglian

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Where did we prove the base case?

Therefore, $P(k + 1)$ is true, completing the induction. ■

When writing a proof by induction,

make sure to prove the base case!

Otherwise, your argument is invalid!

Why did this work?

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2

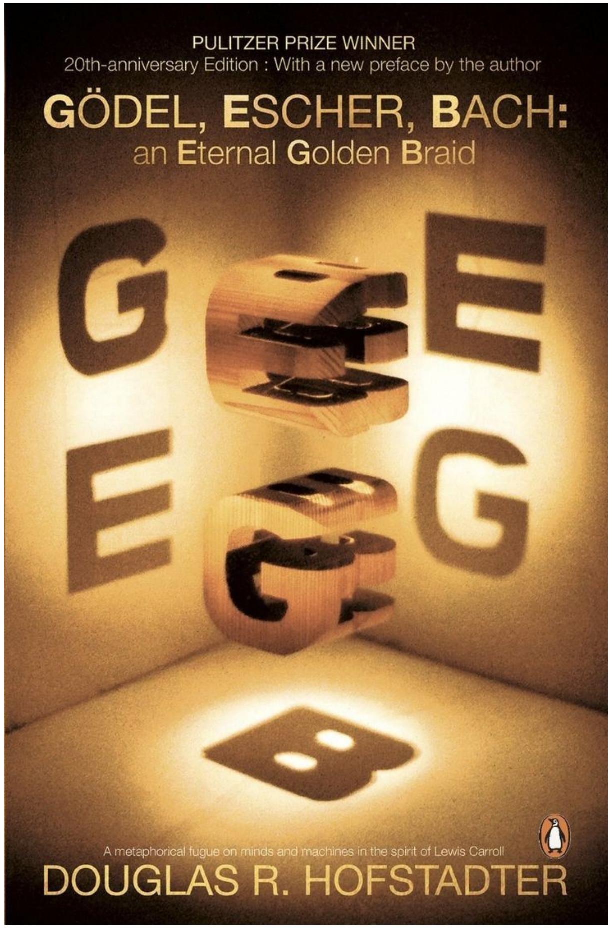
Ther

You can prove ***anything*** from a faulty assumption. This is called the **principle of explosion**. To see why, read "Animal, Vegetable, or Minister" for a silly example.

The MU Puzzle

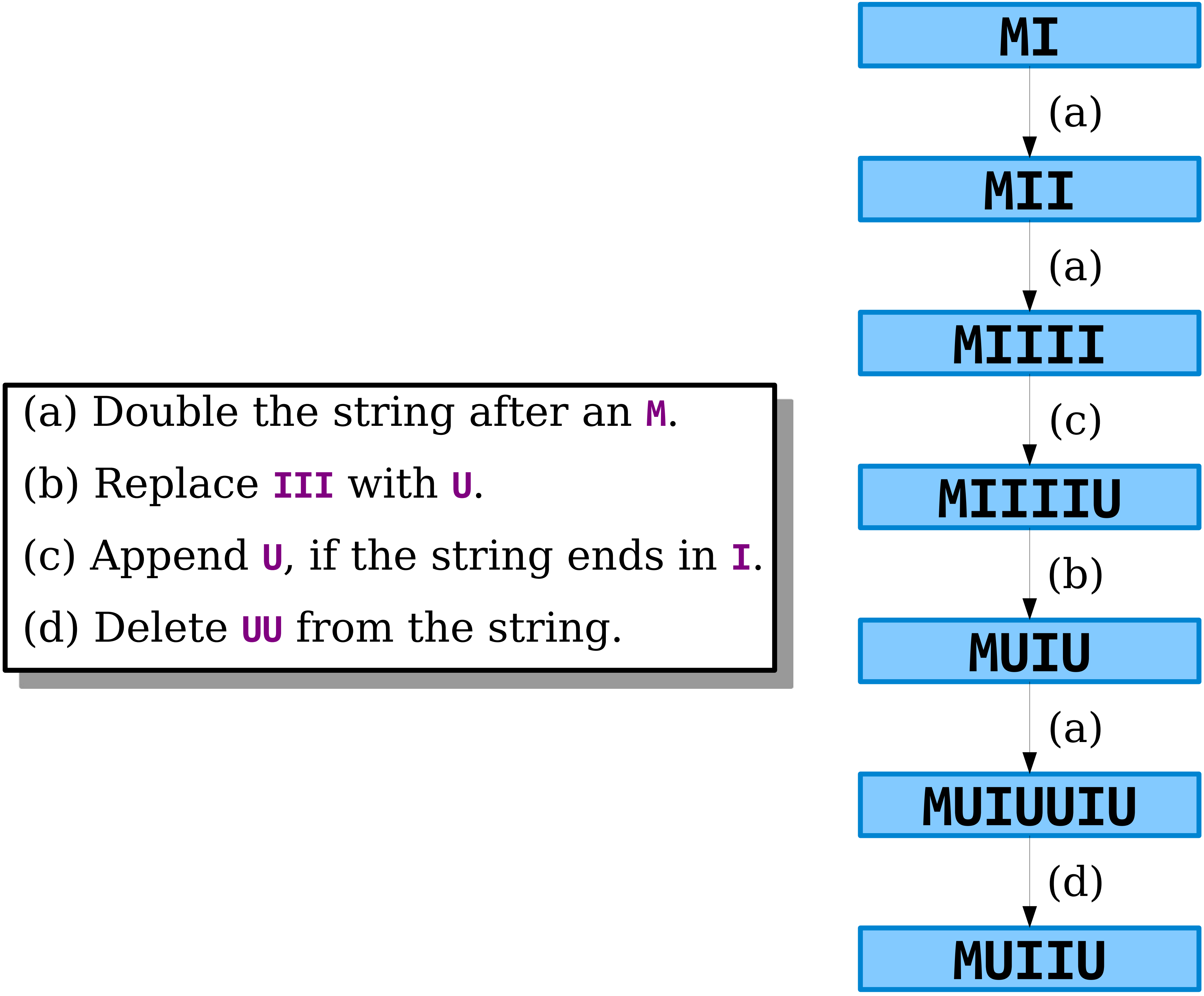
Gödel, Escher Bach: An Eternal Golden Braid

- Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.
- It's a great read after you've finished CS103 – you'll see so many of the ideas we'll cover presented in a totally different way!



The `MU` Puzzle

- Begin with the string `MI`.
- Repeatedly apply one of the following operations:
  - Double the contents of the string after the `M`: for example, `MIIU` becomes `MIIUIIU`, or `MI` becomes `MII`.
  - Replace `III` with `U`: `MIIII` becomes `MUI` or `MIU`.
  - Append `U` to the string if it ends in `I`: `MI` becomes `MIU`.
  - Remove any `UU`: `MUUU` becomes `MU`.
- ***Question***: How do you transform `MI` to `MU`?



(a) Double the string after an M.
(b) Replace III with U.
(c) Append U, if the string ends in I.
(d) Delete UU from the string.
MI
(a)
MII
(a)
MIIII
(c)
MIIIIU
(b)
MUIU
(a)
MUIUUIU
(d)
MUIIU

Try It!

Starting with `MI`, apply these operations to make `MU`:

(a) Double the string after an `M`.

(b) Replace `III` with `U`.

(c) Append `U`, if the string ends in `I`.

(d) Delete `UU` from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of **`I`**'s is *not* a multiple of three.
- To make **`MU`**, the number of **`I`**'s must end up as a multiple of three.
- Can we *ever* make the number of **`I`**'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

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Proof: By contrapositive; we'll prove that if $n - 3$ is a multiple of three, then n is also a multiple of three. Because $n - 3$ is a multiple of three, we can write $n - 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

Lemma: No matter which moves are made, the number of `I`'s in the string never becomes multiple of three.

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Proof: Let $P(n)$ be the statement "after any n moves, the number of `I`'s in the string will not be multiple of three." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

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As a base case, we'll prove $P(0)$, that the number of `I`'s after 0 moves is not a multiple of three. After no moves, the string is `MI`, which has one `I` in it. Since one isn't a multiple of three, $P(0)$ is true.

Lemma: No matter which moves are made, the number of `I`'s in the string never becomes multiple of three.

Proof: Let $P(n)$ be the statement "after any n moves, the number of `I`'s in the string will not be multiple of three." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

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Case 1: Double the string after the **`M`**.

Case 2: Replace **`III`** with **`U`**.

Case 3: Either append **`U`** or delete **`UU`**.

Lemma: No matter which moves are made, the number of **I**'s in the string never becomes multiple of three.

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Therefore, no sequence of $k+1$ moves ends with a multiple of three `I`'s.

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Therefore, no sequence of $k+1$ moves ends with a multiple of three **`I`**'s. Thus $P(k+1)$ is true, completing the induction.

Lemma: No matter which moves are made, the number of **`I`**'s in the string never becomes multiple of three.

Proof: Let $P(n)$ be the statement "after any n moves, the number of **`I`**'s in the string will not be multiple of three." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

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Therefore, no sequence of $k+1$ moves ends with a multiple of three **`I`**'s. Thus $P(k+1)$ is true, completing the induction. ■

Theorem: The **MU** puzzle has no solution.

Proof: Assume for the sake of contradiction that the **MU** puzzle has a solution and that we can convert **MI** to **MU**. This would mean that at the very end, the number of **I**'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of **I**'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the **MU** puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

How Not To Induct, Part 2

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

All Horses are the Same Color

$P(0)$ = "All groups of 0 horses always have the same color"

Vacuously true!

Base case: $n = 0$

All Horses are the Same Color

Assume $P(k)$ = "All groups of k horses always have the same color"

Inductive hypothesis: $n = k$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of k+1 horses always have the same color"

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

By $P(k)$, these k horses have the same color

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

By $P(k)$, these k horses have the same color

By $P(k)$, these k horses have the same color

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets

And we said that both horses on the ends are the same color as these overlapping horses

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

So all $k+1$ horses have the same color!

Inductive hypothesis: $n = k+1$

⚠ Incorrect! ⚠ Proof: Let $P(n)$ be the statement "all groups of n horses are the same color." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

What's wrong with this proof?

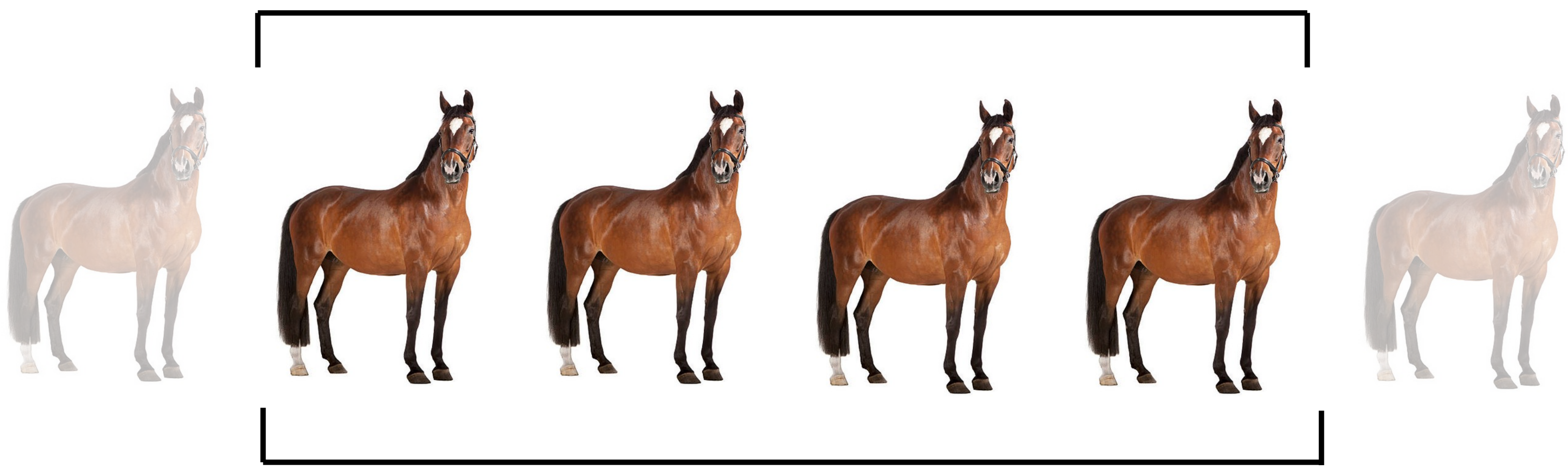
Answer at pollev.com/zhenglian

What’s going on here?

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets



Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets

But what if there are no such horses?

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

$$P(1) \to P(2)$$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

By $P(1)$, this 1 horse has the same color

$$P(1) \rightarrow P(2)$$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

By $P(1)$, this 1 horse has the same color

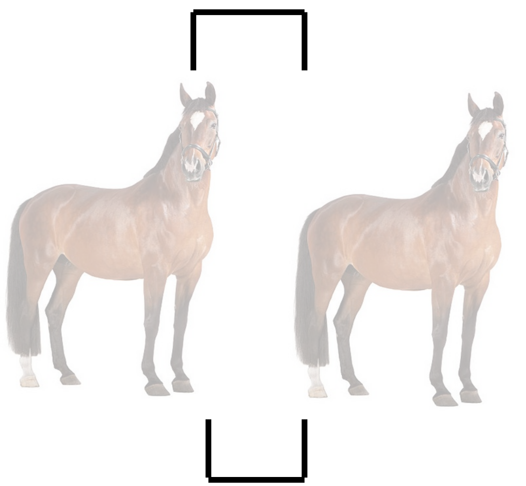
By $P(1)$, this 1 horse has the same color

$$P(1) \rightarrow P(2)$$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

These horses in the middle (??) were in both sets



$$P(1) \to P(2)$$

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For the inducti … ral number k
that $P(k)$ is true … me color. Now
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horses. Again w … se horses are
the same color.

The logic in our inductive step does not allow us to get from $P(1)$ to $P(2)$.
Specifically, there are no non-excluded horses that were in both sets.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. <u>Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color.</u> Thus $P(k+1)$ holds, completing the induction. ■

Non-Issues with this Proof

- *“We should have proven additional base cases”*
  - A proof by induction only needs a single base case, so the fact that we only have one here is not in itself an issue.
- *“We should have used complete induction”*
  - Complete induction wouldn’t have helped us here either, since our inductive step would still need to use $P(0)$ and $P(1)$ to prove $P(2)$.

Induction Debugging Tips

- Remember that induction requires two parts: the base case and the inductive step
- If you see an induction proof of a false statement, one of these pieces must be broken
- Recommendation: try playing the induction out one step at a time (Is the base case true? From the base case, does the reasoning in your inductive step allow you to conclude the next statement? What about the following statement? etc…)

Next Time

- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.